

LOG FANO VARIETIES OVER FUNCTION FIELDS OF CURVES

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ABSTRACT. Consider a smooth log Fano variety over the function field of a curve. Suppose that the boundary has positive normal bundle. Choose an integral model over the curve. Then integral points are Zariski dense, after removing an explicit finite set of points on the base curve.

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1. INTRODUCTION

Let k be an algebraically closed field of characteristic zero, B a smooth projective curve over k with function field $F = k(B)$.

Our point of departure is the following theorem, combining work of Graber-Harris-Starr and Kollar-Miyaoka-Mori [4, 10]: Let X be a smooth projective rationally-connected variety over F . Then $X(F)$ is Zariski dense in X . One central example is Fano varieties, i.e., varieties with ample anticanonical class, which are known to be rationally connected (see [9, V.2.13]). In this context, it is not necessary to pass to a field extension to get rational points.

When F is a number field, it may be necessary to pass to a finite extension to get rational points; there exist Fano varieties over \mathbb{Q} without rational points. Moreover, even for Fano threefolds potential Zariski

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density, i.e., density after a finite extension of F , is unknown in general. For some positive results in this direction see [5], [3] and [6].

In this paper we study Zariski density of *integral* points. Consider pairs (X, D) consisting of a variety X and a divisor $D \subset X$, and fix integral models $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow B$ (see Section 2 for the definition). An F -rational point $s \in X \setminus D$ gives rise to a section $s : B \rightarrow \mathcal{X}$ of π , meeting \mathcal{D} in finitely many points. As we vary s ,

$$s^{-1}(\mathcal{D}) = \pi(s(B) \cap \mathcal{D}) \subset B$$

may vary as well. Fixing a finite set $S \subset B$, an S -integral point of $(\mathcal{X}, \mathcal{D})$ is an F -rational point of X such that $s^{-1}(\mathcal{D}) \subset S$ (as sets).

Theorem 1. *Let F be the function field of smooth projective curve B/k . Let (X, D) be a pair consisting of a smooth projective variety X and a smooth divisor $D \subset X$, defined over F . Assume that $-(K_X + D)$ is ample and the first Chern class $c_1(\mathcal{N}_{D/X})$ is effective and nonzero. Choose an integral model $(\mathcal{X}, \mathcal{D}) \rightarrow B$. Then there is an explicit finite set $S \subset B$ such that S -integral points of $(\mathcal{X}, \mathcal{D})$ are Zariski dense.*

Theorem 7 makes precise how S is chosen.

This is a partial converse to the function-field version of Vojta's conjectures: Integral points are not Zariski dense when the log canonical class $K_X + D$ is ample (see [12] for the number-field case, with connections to value-distribution theory). Very few density results for integral points over number fields are available, most of them in dimension two (see [11], [2], [7]).

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2. INTEGRAL MODELS AND STATEMENTS OF RESULTS

Definition 2. A pair (X, D) consists of a smooth projective variety and a reduced effective divisor with normal crossings.

Let B be a smooth projective curve defined over an algebraically closed field k of characteristic zero and $F = k(B)$ its function field.

Definition 3. Let (X, D) be a pair defined over F . An *integral model*

$$\pi : (\mathcal{X}, \mathcal{D}) \rightarrow B$$

consists of a flat proper morphism from a normal variety $\pi_X : \mathcal{X} \rightarrow B$ with generic fiber X , and a closed subscheme $\mathcal{D} \subset \mathcal{X}$ such that $\pi_D := \pi_X|_{\mathcal{D}} : \mathcal{D} \rightarrow B$ is flat and has generic fiber D .

We emphasize that \mathcal{D} has no irreducible components contained in fibers of π_X .

For many applications, the model is dictated by the specific circumstances. Given an embedding of (X, D) in projective space there is a natural choice of model: The properness of the Hilbert scheme yields extensions of X and D to schemes flat and projective over B . Locally on B , these are obtained by ‘clearing denominators’ in the defining equations of X and D . Normalizing if necessary, we obtain a model of (X, D) .

Definition 4. Let S be a finite subset of B . An *S -integral point of $(\mathcal{X}, \mathcal{D})$* is a section $s : B \rightarrow \mathcal{X}$ such that $s^{-1}(\mathcal{D}) \subset S$ as sets.

Thus if $D = \emptyset$ then integral points are just sections of $\mathcal{X} \rightarrow B$, which are F -rational points of X .

The following proposition is straightforward:

Proposition 5. Let $(\mathcal{X}_1, \mathcal{D}_1)$ and $(\mathcal{X}_2, \mathcal{D}_2)$ be integral models of (X, D) . Let $T \subset B$ denote the set over which the birational map

$$(\mathcal{X}_1, \mathcal{D}_1) \dashrightarrow (\mathcal{X}_2, \mathcal{D}_2)$$

fails to be an isomorphism. S -integral points of $(\mathcal{X}_1, \mathcal{D}_1)$ are mapped to $(S \cup T)$ -integral points of $(\mathcal{X}_2, \mathcal{D}_2)$. If S -integral points of $(\mathcal{X}_1, \mathcal{D}_1)$ are Zariski dense then $(S \cup T)$ -integral points of $(\mathcal{X}_2, \mathcal{D}_2)$ are Zariski dense.

In particular, if we allow ourselves to enlarge the set S then Zariski-density of integral points is independent of the model.

Definition 6. A point $b \in B$ is of *good reduction* if the fibers $\mathcal{X}_b = \pi_X^{-1}(b)$ and $\mathcal{D}_b = \pi_D^{-1}(b)$ are smooth.

Theorem 7. Let (X, D) be a pair over $F = k(B)$ satisfying the following:

- D is smooth and rationally connected;
- the normal bundle $\mathcal{N}_{D/X}$ is effective and nontrivial.

Given a model $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow B$, let S be a nonempty finite set of points in B containing the image of the singular locus of $(\mathcal{X}, \mathcal{D})$. Then S -integral points of $(\mathcal{X}, \mathcal{D})$ are Zariski dense.

Note however that we allow points of bad reduction outside S . For example, let $\mathcal{X} = \mathbb{P}_{[x,y,z]}^2 \times \mathbb{P}_{[s,t]}^1$ and

$$\mathcal{D} = \{s(x^2 + yz) + t(y^2 + xz) = 0\}.$$

The model $(\mathcal{X}, \mathcal{D})$ is smooth but $\mathcal{D}_{[s,t]}$ is singular when $s^3 + t^3 = 0$.

Let K_X denote the canonical class of X and $K_X + D$ the log canonical class of (X, D) . The pair (X, D) is *log Fano* if $-(K_X + D)$ is ample. By adjunction

$$(K_X + D)|_D = K_D$$

so $-K_D$ is ample. Thus D is Fano hence rationally connected [10] [9, V.2.13].

Corollary 8. *Let (X, D) be a log Fano variety over F with X and D smooth. Assume that $\mathcal{N}_{D/X}$ is effective and nontrivial. For each integral model and collection of places as specified in Theorem 7, integral points are Zariski dense.*

We discuss how Theorem 7 can be reduced to the case of nonsingular integral models:

Definition 9. A *good resolution* of an integral model is a birational proper morphism from a pair

$$\rho : (\tilde{\mathcal{X}}, \tilde{\mathcal{D}}) \rightarrow (\mathcal{X}, \mathcal{D})$$

such that

- $\rho^{-1}(\mathcal{D}) = \tilde{\mathcal{D}}$;
- ρ is an isomorphism over the open subset of $(\mathcal{X}, \mathcal{D})$ where \mathcal{X} is smooth and \mathcal{D} is normal crossings.

Remark 10.

- (1) $\tilde{\mathcal{D}}$ may very well have components contained in fibers over B , so $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}})$ is *not* necessarily an integral model.
- (2) The normality assumption guarantees that for each $b \in B$ and each irreducible component of \mathcal{X}_b , the total space \mathcal{X} is smooth at the generic point of that component. In particular, ρ is an isomorphism over a dense open subset of each fiber.

Let $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}) \rightarrow (\mathcal{X}, \mathcal{D})$ be a good resolution, $S \subset B$ a finite set containing the images of the singularities of \mathcal{X} and \mathcal{D} , and $\tilde{\mathcal{D}}^\circ$ the union of the components of $\tilde{\mathcal{D}}$ dominating B . We have:

- $\tilde{\mathcal{D}}^\circ$ is normal crossings;

- images of S -integral points of $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}^\circ)$ under ρ are S -integral points of $(\mathcal{X}, \mathcal{D})$.

We have a bijection

$$\rho : \tilde{\mathcal{X}} \setminus \tilde{\mathcal{D}} \rightarrow \mathcal{X} \setminus \mathcal{D},$$

so S -integral points of $(\mathcal{X}, \mathcal{D})$ correspond to sections

$$\{\tilde{s} : B \rightarrow \tilde{\mathcal{X}} : \tilde{s}^{-1}(\tilde{\mathcal{D}}) \subset S\}.$$

Since the fibral components of $\tilde{\mathcal{D}}$ lie over S , S -integral points of $(\mathcal{X}, \mathcal{D})$ are equal to S -integral points of $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}^\circ)$.

This analysis reduces Theorem 7 to:

Theorem 11 (Smooth case). *Retain the assumptions of Theorem 7 and assume in addition that \mathcal{X} and \mathcal{D} are nonsingular. Then for any nonempty $S \subset B$ the S -integral points in $(\mathcal{X}, \mathcal{D})$ are Zariski dense.*

3. ATIYAH CLASSES AND FREE CURVES

We work over an algebraically closed field k .

Let C be a smooth projective variety with tangent sheaf \mathcal{T}_C . Its deformation space is denoted $\text{Def}(C)$ and first-order deformations are given by $H^1(C, \mathcal{T}_C)$. Let L be an invertible sheaf on C and $p : L \rightarrow C$ the line bundle defined by the same cocycle. The deformation space of L is denoted $\text{Def}(L)$ and first-order deformations are given by $H^1(C, \mathcal{O}_C)$.

Let $\text{Def}(C, L)$ denote deformations of both C and L . Taking \mathbb{G}_m -invariants of the tangent-bundle exact sequence

$$0 \rightarrow \mathcal{T}_{L/C} \rightarrow \mathcal{T}_L \rightarrow p^*\mathcal{T}_C \rightarrow 0$$

we obtain the Atiyah extension [1]

$$(3.1) \quad 0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{E}_{C,L} \rightarrow \mathcal{T}_C \rightarrow 0.$$

This is classified by an element $\lambda \in \text{Ext}^1(\mathcal{T}_C, \mathcal{O}_C) = H^1(C, \Omega_C^1)$, which (up to sign) agrees with the Chern class $c_1(L)$ [1, pp. 196]. First-order deformations of (C, L) are given by $H^1(C, \mathcal{E}_{C,L})$ [8, pp. 241]. The homomorphisms in the long exact sequence

$$H^1(C, \mathcal{O}_C) \rightarrow H^1(C, \mathcal{E}_{C,L}) \rightarrow H^1(C, \mathcal{T}_C)$$

are the differentials of natural morphisms of deformation spaces

$$\text{Def}(L) \rightarrow \text{Def}(C, L) \rightarrow \text{Def}(C).$$

Consider the case where C is a reduced projective scheme, perhaps with singularities. First-order deformations are given by $\text{Ext}^1(\Omega_C^1, \mathcal{O}_C)$.

Working directly with Kähler differentials rather than tangent bundles, we obtain a dual version of the Atiyah extension [8, pp. 241]

$$(3.2) \quad 0 \rightarrow \Omega_C^1 \rightarrow \mathcal{F}_{C,L} \rightarrow \mathcal{O}_C \rightarrow 0,$$

again classified by an element $\lambda = \pm c_1(L) \in H^1(C, \Omega_C^1)$. The long exact sequence

$$\mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) \rightarrow \mathrm{Ext}^1(\mathcal{F}_{C,L}, \mathcal{O}_C) \rightarrow \mathrm{Ext}^1(\Omega_C^1, \mathcal{O}_C)$$

gives the differentials of

$$\mathrm{Def}(L) \rightarrow \mathrm{Def}(C, L) \rightarrow \mathrm{Def}(C).$$

Now let C be a nodal curve embedded in a smooth projective variety Y ; the component of the Hilbert scheme parametrizing deformations of C in Y is denoted $\mathrm{Def}(C \subset Y)$. First-order deformations of $C \subset Y$ correspond to

$$\mathrm{Hom}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C) = \Gamma(C, \mathcal{N}_{C/Y});$$

here \mathcal{I}_C is the ideal sheaf and $\mathcal{N}_{C/Y} = \mathrm{Hom}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$ the normal sheaf. Let $\mathrm{Def}(C \rightarrow Y)$ denote deformations of the *morphism* $C \rightarrow Y$; first-order deformations correspond to $\mathrm{Hom}(\Omega_Y^1|C, \mathcal{O}_C) = \Gamma(C, T_Y|C)$ (see [9, I.2]).

Fix a line bundle L on Y . Consider the morphisms

$$\begin{aligned} \mu : \mathrm{Def}(C \subset Y) &\rightarrow \mathrm{Def}(C, L|C) \\ \{C' \subset Y\} &\mapsto (C', L|C') \end{aligned}$$

and

$$\begin{aligned} \nu : \mathrm{Def}(C \rightarrow Y) &\rightarrow \mathrm{Def}(L|C) \\ \{f : C' \rightarrow Y\} &\mapsto f^*L. \end{aligned}$$

The same deformation space parametrizes fibers of both μ and ν :

$$(3.3) \quad \mathrm{Def}((C, L|C) \rightarrow (Y, L)) = \{(f, M, \alpha) : f \in \mathrm{Def}(C \rightarrow Y), \\ M \in \mathrm{Def}(L|C), \alpha : M \xrightarrow{\sim} f^*L\}.$$

This has tangent space $\text{Hom}(\mathcal{F}_{Y,L}|C, \mathcal{O}_C) = \Gamma(C, \mathcal{E}_{Y,L}|C)$ and obstruction space $\text{Ext}^1(\mathcal{F}_{Y,L}|C, \mathcal{O}_C) = H^1(C, \mathcal{E}_{Y,L}|C)$. Indeed, we have the diagram

$$(3.4) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{I}_C/\mathcal{I}_C^2 & = & \mathcal{I}_C/\mathcal{I}_C^2 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \Omega_Y^1|C & \rightarrow & \mathcal{F}_{Y,L}|C & \rightarrow & \mathcal{O}_C \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & \Omega_C^1 & \rightarrow & \mathcal{F}_{C,L|C} & \rightarrow & \mathcal{O}_C \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

where the middle row is obtained by restricting the dual Atiyah extension of (Y, L) to C . Applying $\text{Hom}(-, \mathcal{O}_C)$ to this diagram, we obtain differentials between the various deformation spaces we have introduced. The long exact sequence arising from the second row is

$$0 \longrightarrow \Gamma(\mathcal{O}_C) \longrightarrow \Gamma(\mathcal{E}_{Y,L}|C) \longrightarrow \Gamma(T_Y|C) \xrightarrow{d\nu} H^1(\mathcal{O}_C);$$

the second column yields

$$0 \longrightarrow \Gamma(\mathcal{E}_{C,L|C}) \longrightarrow \Gamma(\mathcal{E}_{Y,L}|C) \longrightarrow \Gamma(\mathcal{N}_{C/Y}) \xrightarrow{d\mu} H^1(\mathcal{E}_{C,L|C}).$$

A morphism is guaranteed to be smooth when its fibers are unobstructed (cf. [9, I.2.17.2]), thus we have

Proposition 12. *Let C be a nodal curve embedded in a smooth projective variety Y . If $H^1(C, \mathcal{E}_{Y,L}|C) = 0$ then μ is smooth at $C \subset Y$ and ν is smooth at $C \rightarrow Y$.*

Definition 13. Let Y be a smooth projective variety with line bundle L and C a nodal projective curve. A nonconstant morphism $f : C \rightarrow Y$ is L -free if for each $q \in C$

$$H^1(C, f^*\mathcal{E}_{Y,L} \otimes \mathcal{I}_q) = 0.$$

It is L -very free if for each subscheme $\Sigma \subset C$ of length two

$$H^1(C, f^*\mathcal{E}_{Y,L} \otimes \mathcal{I}_\Sigma) = 0.$$

Any L -free (resp. very free) morphism is free (resp. very free) as \mathcal{T}_Y is a quotient of $\mathcal{E}_{Y,L}$.

We now assume that k is of characteristic zero.

Proposition 14. *Let Y be a smooth rationally connected projective variety, L a line bundle on Y , and $y \in Y$. Then Y admits an L -free morphism $f : \mathbb{P}^1 \rightarrow Y$ with image containing y . If L is effective and nontrivial then f can be chosen to be L -very free.*

Proof. There exists a very free morphism $g : \mathbb{P}^1 \rightarrow Y$ [9, IV.3.9.4]; moreover, given any finite collection of points $y_1, \dots, y_m \in Y$, we may assume the image of g contains these points.

We have the extension

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow g^* \mathcal{E}_{Y,L} \rightarrow g^* \mathcal{T}_Y \rightarrow 0$$

where $g^* \mathcal{T}_Y$ is ample. It follows that each summand of $g^* \mathcal{E}_{Y,L}$ is nonnegative, which implies L -freeness.

Now assume H is an effective nonzero divisor corresponding to L . Choose g such that its image contains y , some point y' in the support of H , and some point y'' not in the support of H . In particular, the image is not contained in any component of H . It follows that $g^* L$ has positive degree.

If $\mathcal{O}_{\mathbb{P}^1}$ were a summand of $g^* \mathcal{E}_{Y,L}$ then we would have

$$g^* \mathcal{E}_{Y,L} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus g^* \mathcal{T}_Y,$$

i.e., the Atiyah extension would split after pull-back. The extension induced by Diagram 3.4

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{E}_{\mathbb{P}^1, g^* L} \rightarrow \mathcal{T}_{\mathbb{P}^1} \rightarrow 0$$

would split as well. However, this extension is classified by

$$\pm c_1(g^* L) \in H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1) = \text{Ext}^1(\mathcal{T}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}),$$

which is nontrivial. □

A *comb with broken teeth* is a nodal projective curve

$$C = B \cup T_1 \cup \dots \cup T_r$$

where B is smooth and each T_i is a tree of smooth rational curves meeting B in a point $q_i \in B$. Let $\sigma : C \rightarrow B$ denote the morphism which is the identity on B and which contracts each T_i to q_i . There is a natural stabilization morphism

$$\text{stab} : \text{Def}(C) \rightarrow \text{Def}(B)$$

constructed as follows: Pick points

$$b_1, \dots, b_n \in B \setminus \{q_1, \dots, q_r\}$$

with $2g(B) - 2 + n > 0$, so that (B, b_1, \dots, b_n) is a stable pointed curve. The Knudsen-Mumford stabilization of $(C, \sigma^{-1}(b_1), \dots, \sigma^{-1}(b_n))$ is (B, b_1, \dots, b_n) . Each deformation of C arises from a deformation of $(C, \sigma^{-1}(b_1), \dots, \sigma^{-1}(b_n))$. Thus the stabilization morphism

$$\text{Def}(C, \sigma^{-1}(b_1), \dots, \sigma^{-1}(b_n)) \rightarrow \overline{M}_{g(B), n}$$

induces stab .

For each $e \in H^2(C, \mathbb{Z})$ (resp. $d \in H^2(B, \mathbb{Z})$), let $\text{Pic}^e(C)$ (resp. $\text{Pic}^d(B)$) denote the corresponding component of the Picard scheme. We have a morphism

$$\begin{aligned} \sigma_* : \text{Pic}(C) &\rightarrow \text{Pic}(B) \\ M &\mapsto M|B \otimes \mathcal{O}_B(e_1 q_1 + \dots + e_r q_r), \quad e_i = \deg(M|T_i) \end{aligned}$$

mapping each component $\text{Pic}^e(C)$ isomorphically onto the component $\text{Pic}^d(B)$ containing its image. Thus we get a morphism

$$\text{stab}' : \text{Def}(C, M) \rightarrow \text{Def}(B, \sigma_* M)$$

and a commutative diagram

$$\begin{array}{ccc} \text{Def}(C, M) & \xrightarrow{\text{stab}'} & \text{Def}(B, \sigma_* M) \\ \downarrow & & \downarrow \\ \text{Def}(C) & \xrightarrow{\text{stab}} & \text{Def}(B). \end{array}$$

Consider the composition

$$(3.5) \quad \tau : \text{Def}(C \subset Y) \xrightarrow{\mu} \text{Def}(C, L|C) \xrightarrow{\text{stab}'} \text{Def}(B, \sigma_*(L|C)).$$

A fiber of τ corresponds to deformations of C which do not affect the line bundle induced by push-forward to the stabilization.

Proposition 15. *Let B be a smooth projective curve embedded in a smooth variety Y and L a line bundle on Y . Assume that $H^1(B, \mathcal{E}_{Y,L}|B) = 0$ and consider a comb*

$$C = B \cup T_1 \cup \dots \cup T_r$$

such that the T_i are L -free curves on Y . Then the fiber $\tau^{-1}(\tau(C))$ contains a smoothing of C .

Proof. Write $q_i = B \cap T_i$ so that

$$H^1(T_i, \mathcal{E}_{Y,L} \otimes \mathcal{O}_{T_i}(-q_i)) = 0.$$

Our vanishing assumption and an induction on the number of components imply (see [9, II.7.5]) $H^1(C, \mathcal{E}_{Y,L}|C) = 0$.

We describe a flat morphism $\varpi : \mathcal{C} \rightarrow \mathbb{A}^r$ deforming C to B (see [9, pp. 156]). Consider the smooth codimension-two subvariety

$$Z = \cup_{i=1}^r (\{q_i\} \times \{t_i = 0\}) \subset B \times \mathbb{A}^r$$

and the blow-up

$$\sigma : \mathcal{C} := \text{Bl}_Z(B \times \mathbb{A}^r) \rightarrow B \times \mathbb{A}^r$$

with exceptional divisors E_1, \dots, E_r . The composed morphism

$$\varpi : \mathcal{C} \rightarrow B \times \mathbb{A}^r \rightarrow \mathbb{A}^r$$

is still flat with $\varpi^{-1}(0) = C$; every fiber of ϖ is a comb with handle B and the blow-down map is the stabilization contraction relative to \mathbb{A}^r . We introduce a line bundle on this family: Consider

$$L' = L|B \otimes \mathcal{O}_B(e_1 q_1 + \dots + e_r q_r)$$

where $e_i = L \cdot T_i$ and write

$$\mathcal{M} = (\pi_B \circ \sigma)^* L' \otimes \mathcal{O}_{\mathcal{C}}(-e_1 E_1 - \dots - e_r E_r).$$

This is chosen so that

$$\mathcal{M}|_{\varpi^{-1}(0)} = \mathcal{M}|C = L|C$$

and $\sigma_* \mathcal{M} = \pi_B^* L'$.

We state a relative version of the deformation space 3.3: Consider morphisms

$$f : \mathcal{C} \rightarrow Y \times \mathbb{A}^r$$

over \mathbb{A}^r admitting an isomorphism $\alpha : \mathcal{M} \rightarrow f^*(\pi_Y^* L)$. This is represented by a scheme

$$\text{Def}((\mathcal{C}, \mathcal{M}) \rightarrow (Y \times \mathbb{A}^r, \pi_Y^* L)) \rightarrow \mathbb{A}^r$$

over \mathbb{A}^r . The vanishing $H^1(C, \mathcal{E}_{Y,L}|C) = 0$ shows this problem is unobstructed over $0 \in \mathbb{A}^r$ and thus the deformation space is smooth over a neighborhood of $0 \in \mathbb{A}^r$. In particular, it contains a smoothing of C to B .

By construction, the image of

$$\text{Def}((\mathcal{C}, \mathcal{M}) \rightarrow (Y \times \mathbb{A}^r, \pi_Y^* L)) \rightarrow \text{Def}(C \subset Y)$$

is contained in the fiber of τ . \square

The vanishing condition of Proposition 15 also guarantees that μ is smooth (see Proposition 12). We indicate how to achieve this in practice

Proposition 16. *Let Y be a smooth projective rationally connected variety, L an effective nontrivial line bundle, and B a smooth proper curve embedded in Y . Then there exists a comb*

$$C = B \cup T_1 \cup \dots \cup T_r$$

such that C deforms to a smooth L -free curve. In particular, τ is dominant at $(C \subset Y) \in \text{Def}(C \subset Y)$.

Proof. Proposition 14 gives L -very free rational curves through each point of Y . We use these to construct a comb with handle B and $n \gg 0$ L -very free teeth T_1, \dots, T_n . The Hard Smoothing technique of [9, II.7.10] implies that a subcomb

$$B \cup T_{i_1} \cup \dots \cup T_{i_r}$$

deforms to a smooth L -free curve.

Recall that $\tau = \text{stab}' \circ \mu$ (see (3.5)) and μ is smooth near C . Since stab' is birational, τ is dominant. \square

4. RELATIVE ATIYAH CLASSES AND FREE CURVES

In this section we work over an algebraically closed field of characteristic zero. See [8] for general background on relative obstruction theory.

Let B be a smooth projective curve. Fix a proper nodal curve C over B . Let $\text{Def}(C/B)$ denote deformations of C over B ; first-order deformations are parametrized by $\text{Ext}^1(\Omega_{C/B}^1, \mathcal{O}_C)$. If L is a line bundle on C then we have the relative Atiyah extension

$$0 \rightarrow \Omega_{C/B}^1 \rightarrow \mathcal{F}_{C,L/B} \rightarrow \mathcal{O}_C \rightarrow 0.$$

Let $\text{Def}(C, L/B)$ denote deformations of C and L over B ; $\text{Ext}^1(\mathcal{F}_{C,L/B}, \mathcal{O}_C)$ parametrizes the first-order deformations.

Let $\pi : \mathcal{Y} \rightarrow B$ be a nonconstant proper morphism from a smooth variety. Given an embedding $C \subset \mathcal{Y}$ over B , we consider the deformation space

$$\text{Def}(C \rightarrow \mathcal{Y}/B)$$

parametrizing deformations of the map $C \rightarrow \mathcal{Y}$ over B . This has tangent space $\Gamma(C, \mathcal{T}_{\mathcal{Y}/B}|C)$ and obstruction space $H^1(C, \mathcal{T}_{\mathcal{Y}/B}|C)$. Deformations of C as a subscheme of \mathcal{Y} are the same as deformations of C as a subscheme of \mathcal{Y} over B , i.e.,

$$\text{Def}(C \subset \mathcal{Y}/B) = \text{Def}(C \subset \mathcal{Y}).$$

Let \mathcal{L} be an invertible sheaf on \mathcal{Y} such that the restriction to the generic fiber of π is effective and nonzero. We have a relative version of the Atiyah extension

$$0 \rightarrow \mathcal{O}_{\mathcal{Y}} \rightarrow \mathcal{E}_{\mathcal{Y}, \mathcal{L}/B} \rightarrow \mathcal{T}_{\mathcal{Y}/B} \rightarrow 0,$$

classified by the image of the ordinary Atiyah class under the restriction

$$\mathrm{Ext}^1(\mathcal{T}_{\mathcal{Y}}, \mathcal{O}_Y) \rightarrow \mathrm{Ext}^1(\mathcal{T}_{\mathcal{Y}/B}, \mathcal{O}_Y).$$

Our effectiveness assumption guarantees the extension is not split.

Consider the morphisms

$$\mu_{/B} : \mathrm{Def}(C \subset Y) \rightarrow \mathrm{Def}(C, L|C/B)$$

and

$$\nu_{/B} : \mathrm{Def}(C \rightarrow Y/B) \rightarrow \mathrm{Def}(L|C),$$

i.e., the relative versions of μ and ν defined above. Their fibers are given by the relative version of (3.3):

$$(4.1) \quad \begin{aligned} \mathrm{Def}((C, \mathcal{L}|C) \rightarrow (\mathcal{Y}, \mathcal{L})/B) &= \{(f, M, \alpha) : f \in \mathrm{Def}(C \rightarrow \mathcal{Y}/B), \\ &\quad M \in \mathrm{Def}(\mathcal{L}|C), \alpha : M \xrightarrow{\sim} f^*\mathcal{L}\}, \end{aligned}$$

with tangent space $\Gamma(C, \mathcal{E}_{\mathcal{Y}, \mathcal{L}/B}|C)$ and obstruction space $H^1(C, \mathcal{E}_{\mathcal{Y}, \mathcal{L}/B}|C)$. Here $\mathcal{E}_{\mathcal{Y}, \mathcal{L}/B} = \mathcal{F}_{\mathcal{Y}, \mathcal{L}/B}^*$ where $\mathcal{F}_{\mathcal{Y}, \mathcal{L}/B}$ is defined by the relative analog of Diagram 3.4:

$$(4.2) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{I}_C/\mathcal{I}_C^2 & = & \mathcal{I}_C/\mathcal{I}_C^2 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \Omega_{\mathcal{Y}/B}^1|C & \rightarrow & \mathcal{F}_{\mathcal{Y}, \mathcal{L}/B}|C & \rightarrow & \mathcal{O}_C \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & \Omega_{C/B}^1 & \rightarrow & \mathcal{F}_{C, \mathcal{L}|C/B} & \rightarrow & \mathcal{O}_C \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Just as before, we obtain:

Proposition 17. *Retain the notation introduced above and assume that $H^1(C, \mathcal{E}_{Y, L/B}|C) = 0$. Then the morphisms $\mu_{/B}$ and $\nu_{/B}$ are smooth at $C \subset \mathcal{Y}$.*

Definition 18. A nonconstant morphism from a nodal curve $f : C \rightarrow \mathcal{Y}$ is *free over B* if for each $q \in C$

$$H^1(C, f^*\mathcal{T}_{\mathcal{Y}/B} \otimes \mathcal{I}_q) = 0.$$

It is *very free over B* if for each subscheme $\Sigma \subset C$ of length two

$$H^1(C, f^*\mathcal{T}_{\mathcal{Y}/B} \otimes \mathcal{I}_\Sigma) = 0.$$

It is \mathcal{L} -*free* or \mathcal{L} -*very free over B* if the analogous conditions hold for $\mathcal{E}_{\mathcal{Y}, \mathcal{L}/B}$.

From now on, we will assume that the generic fiber of $\pi : \mathcal{Y} \rightarrow B$ is rationally connected.

Proposition 19. Choose $b \in B$ such that $\mathcal{Y}_b = \pi^{-1}(b)$ is smooth, and $y \in \mathcal{Y}_b$. Then there exists a morphism $f : \mathbb{P}^1 \rightarrow \mathcal{Y}_b$ that is \mathcal{L} -very free over B , with $y \in f(\mathbb{P}^1)$.

Proof. Our blanket assumptions imply $\mathcal{L}|_{\mathcal{Y}_b}$ is nontrivial in every fiber of π . (Nonzero divisors cannot specialize to zero in smooth fibers.) Proposition 14 then gives an $\mathcal{L}|_{\mathcal{Y}_b}$ -very free curve in \mathcal{Y}_b . Since $\mathcal{T}_{\mathcal{Y}/B}|_{\mathcal{Y}_b} = \mathcal{T}_{\mathcal{Y}_b}$, we conclude this curve is \mathcal{L} -very free over B . \square

We will require the following relative version of Proposition 16:

Proposition 20. Let $B' \subset \mathcal{Y}$ be a smooth projective curve, not contained in a singular fiber of π . Then there exists a comb

$$C = B' \cup T_1 \cup \dots \cup T_r$$

with teeth contained in smooth fibers of π , such that C deforms to a smooth curve that is \mathcal{L} -free over B .

Proof. (cf. [4, pp.63]) If B' is contained in a smooth fiber \mathcal{Y}_b of π , we can apply Proposition 16 with $Y = \mathcal{Y}_b$ and $B = B'$. Generally, the same proof applies: Choose \mathcal{L} -very free curves T_1, \dots, T_n in smooth fibers $\mathcal{Y}_{b_1}, \dots, \mathcal{Y}_{b_n}$, meeting B' at points q_1, \dots, q_n . Deformations of the comb

$$B' \cup_{q_1} T_1 \cup_{q_2} T_2 \dots \cup_{q_n} T_n$$

may be obstructed, but some subcomb

$$B' \cup T_{i_1} \cup \dots \cup T_{i_r}$$

will smooth. Again, the method of [9, II.7.10] guarantees that the generic such smoothing is \mathcal{L} -free. \square

Let $C = B' \cup T_1 \cup \dots \cup T_r$ be a comb with handle B' and broken teeth T_1, \dots, T_r contained in fibers $\mathcal{Y}_{b_1}, \dots, \mathcal{Y}_{b_r}$. In analogy to (3.5) we define

$$(4.3) \quad \tau_{/B} : \text{Def}(C \subset \mathcal{Y}) \xrightarrow{\mu_{/B}} \text{Def}(C, \mathcal{L}|C/B) \xrightarrow{\text{stab}'} \text{Def}(B', \sigma_*(\mathcal{L}|C)/B),$$

where $\sigma : C \rightarrow B'$ contracts the teeth of the comb and stab' is the stabilization introduced previously. We can now state the relative formulation of Proposition 15:

Proposition 21. *Retain the notation introduced above and consider a comb*

$$C = B' \cup T_1 \cup \dots \cup T_r$$

such that each T_i is a \mathcal{L} -free curve over B in a smooth fiber \mathcal{Y}_{b_i} . If $H^1(B', \mathcal{E}_{\mathcal{Y}, \mathcal{L}/B}|B') = 0$ then the fiber $\tau_{/B}^{-1}(\tau_{/B}(C))$ contains a smoothing of C .

Remark 22. If the T_i are \mathcal{L} -very free then we may apply Proposition 20 to show the smoothing of C is \mathcal{L} -free over B .

Our main application is to sections of rationally-connected fibrations:

Theorem 23. *Let B be a smooth projective curve, $\pi : \mathcal{Y} \rightarrow B$ a proper morphism from a smooth variety with rationally connected generic fiber, and \mathcal{L} an invertible sheaf on \mathcal{Y} restricting to a nontrivial effective divisor on the generic fiber of π . Fix an integer $N \gg 0$.*

For each invertible sheaf $M \in \text{Pic}^N(B)$, there exists a section $s : B \rightarrow \mathcal{Y}$ such that $s^\mathcal{L} = M$ and s is \mathcal{L} -free over B . In particular, the sheaves*

$$s^*\mathcal{E}_{\mathcal{Y}, \mathcal{L}/B} \text{ and } s^*\mathcal{T}_{\mathcal{Y}/B} = \mathcal{N}_{s(B)/\mathcal{Y}}$$

are both globally generated with no higher cohomology.

In particular, Proposition 12 shows that the morphism

$$\begin{aligned} \mu : \text{Def}(s(B) \subset \mathcal{Y}) &\rightarrow \text{Pic}^N(B) \\ s_t(B) &\mapsto s_t^*\mathcal{L} \end{aligned}$$

is smooth at $s(B)$.

Proof of 23. The Graber-Harris-Starr Theorem [4] gives a section $s_1 : B \rightarrow \mathcal{Y}$. The exact sequence

$$0 \rightarrow \mathcal{T}_{\mathcal{Y}/B} \rightarrow \mathcal{T}_{\mathcal{Y}} \rightarrow \pi^*\mathcal{T}_B \rightarrow 0$$

induces

$$0 \rightarrow s_1^*\mathcal{T}_{\mathcal{Y}/B} \rightarrow s_1^*\mathcal{T}_{\mathcal{Y}} \rightarrow \mathcal{T}_B \rightarrow 0$$

which is split by the differential $ds_1 : \mathcal{T}_B \rightarrow s_1^*\mathcal{T}_{\mathcal{Y}}$. Thus we have

$$s_1^*\mathcal{T}_{\mathcal{Y}} = s_1^*\mathcal{T}_{\mathcal{Y}/B} \oplus \mathcal{T}_B$$

and the first term coincides with the normal bundle $\mathcal{N}_{s_1(B)/\mathcal{Y}}$.

Proposition 20 yields a section $s_2 : B \rightarrow \mathcal{Y}$ that is \mathcal{L} -free over B so that

$$H^1(B, s_2^*\mathcal{E}_{\mathcal{Y}, \mathcal{L}/B}) = 0.$$

To complete the proof, we apply Proposition 21 to produce a smoothing $s : B \rightarrow \mathcal{Y}$ of a comb

$$C = s_2(B) \cup_{b_1} T_1 \dots \cup_{b_r} T_r.$$

However, it is necessary to relate the points of attachment to the precise value of $s^*\mathcal{L}$. Choose e sufficiently large so that for each smooth fiber \mathcal{Y}_b and every point $y \in \mathcal{Y}_b$, there exists an \mathcal{L} -very free curve $T \subset \mathcal{Y}_b$ passing through y with $\mathcal{L} \cdot T = e$. (In Proposition 14, we explained how to ensure that T intersects \mathcal{L} positively.) We therefore may assume

$$e = T_1 \cdot \mathcal{L} = T_2 \cdot \mathcal{L} = \dots = T_r \cdot \mathcal{L} > 0$$

so that

$$\sigma_*(\mathcal{L}|C) = (s_2^*\mathcal{L})(eb_1 + \dots + eb_r).$$

Recall that $\tau_{/B}$ was defined so that for deformations $C' \subset \mathcal{Y}$ in the fiber $\tau_{/B}$ containing $C \subset \mathcal{Y}$, the divisor class $\sigma_*(\mathcal{L}|C') \in \text{Pic}(B)$ remains constant. Thus we have

$$s^*\mathcal{L} = s_2^*\mathcal{L}(eb_1 + \dots + eb_r).$$

Let $U \subset B$ denote the locus where \mathcal{Y}_b is smooth and contains a \mathcal{L} -very free curve T of degree e . It remains to verify the following prime avoidance result, which governs the precise value of N :

Lemma 24. *Let B be a smooth projective curve, $U \subset B$ a dense open subset, and $e \in \mathbb{N}$. Fix a line bundle Λ on B of degree ℓ , $r \geq 2g(B) + 1$, and $N = er + \ell$. For any $M \in \text{Pic}^N(B)$ there exist distinct points $b_1, \dots, b_r \in U$ so that*

$$M \simeq \Lambda(e(b_1 + \dots + b_r)).$$

Proof. This is an elementary application of Riemann-Roch. Choose an e th root of $M \otimes \Lambda^{-1}$, i.e., a line bundle A with $A^{\otimes e} \otimes \Lambda = M$. Any line bundle of degree r on B is very ample so consider the embedding

$$\phi_A : B \hookrightarrow \mathbb{P}^{r-g(B)}.$$

The divisors with any support along $B \setminus U$ form a finite union of hyperplanes in the linear system $|A|$. The divisors admitting points of multiplicity > 1 form a proper subvariety of $\Delta \subset |A|$ by the Bertini Theorem. Any divisor in the complement of the hyperplanes and Δ can be expressed in the form $b_1 + \dots + b_r$ with the b_i distinct in U . \square

This concludes the proof of Theorem 23. \square

5. SMOOTH CASE

In this section, we prove Theorem 11; take $S = \{p\}$ for some $p \in B$.

Apply Theorem 23 to $\mathcal{Y} = \mathcal{D}$, $\mathcal{L} = \mathcal{O}_{\mathcal{D}}(\mathcal{D}) = \mathcal{N}_{\mathcal{D}/X}$, and $M = \mathcal{O}_B(Np)$ for some suitable $N \gg 0$. We obtain a section $s : B \rightarrow \mathcal{D}$ with the following properties:

- $s^*\mathcal{D} = \mathcal{N}_{\mathcal{D}/X}|s(B) \simeq \mathcal{O}_B(Np)$;
- $\mathcal{E}_{\mathcal{D}, \mathcal{O}(\mathcal{D})/B}|s(B)$ is globally generated with no higher cohomology.

Thus Proposition 17 guarantees

$$\mu_{/B} : \text{Def}(s(B) \subset \mathcal{D}) \rightarrow \text{Def}(s(B), \mathcal{O}_{\mathcal{D}}(\mathcal{D})|s(B)/B) \simeq \text{Pic}^N(B)$$

is smooth. Let

$$\text{Def}((B, M) \rightarrow (\mathcal{D}, \mathcal{L})/B) = \text{Def}((B, \mathcal{O}_B(Np)) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}}(\mathcal{D}))/B)$$

denote the fiber, i.e., deformations $s_t : B \rightarrow \mathcal{D}$ such that $s_t^*\mathcal{D} = \mathcal{O}_B(Np)$.

Consider the corresponding deformation space for X

$$\text{Def}((B, \mathcal{O}_B(Np)) \rightarrow (X, \mathcal{O}_X(\mathcal{D}))/B);$$

its obstruction theory is governed by the sheaf $\mathcal{E}_{X, \mathcal{O}(\mathcal{D})/B}|s(B)$. Consider the extensions defining $\mathcal{E}_{\mathcal{D}, \mathcal{O}(\mathcal{D})/B}$ and $\mathcal{E}_{X, \mathcal{O}(\mathcal{D})/B}$, restricted to $s(B)$:

$$\begin{array}{ccccccc} & & 0 & & & 0 & \\ & & \downarrow & & & \downarrow & \\ 0 & \rightarrow & \mathcal{O}_B & \rightarrow & \mathcal{E}_{\mathcal{D}, \mathcal{O}(\mathcal{D})/B}|s(B) & \rightarrow & \mathcal{N}_{s(B)/\mathcal{D}} & \rightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{O}_B & \rightarrow & \mathcal{E}_{X, \mathcal{O}(\mathcal{D})/B}|s(B) & \rightarrow & \mathcal{N}_{s(B)/X} & \rightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{N}_{\mathcal{D}/X}|s(B) & = & \mathcal{N}_{\mathcal{D}/X}|s(B) & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since the terms in the bottom row are isomorphic to $\mathcal{O}_B(Np)$, which has no higher cohomology, we deduce that $\mathcal{E}_{X, \mathcal{O}(\mathcal{D})/B}|s(B)$ has no higher

cohomology. In particular,

$$\mathrm{Def}((B, \mathcal{O}_B(Np)) \rightarrow (\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D}))/B)$$

is unobstructed and smooth.

The inclusion of \mathcal{D} in \mathcal{X} induces an embedding

$$(5.1) \quad \mathrm{Def}((B, \mathcal{O}_B(Np)) \rightarrow (\mathcal{D}, \mathcal{O}_{\mathcal{D}}(\mathcal{D}))/B) \hookrightarrow \mathrm{Def}((B, \mathcal{O}_B(Np)) \rightarrow (\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D}))/B).$$

The image is precisely the indeterminacy of the rational map

$$\begin{aligned} G : \mathrm{Def}((B, \mathcal{O}_B(Np)) \rightarrow (\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D}))/B) &\dashrightarrow \mathbb{P}(\Gamma(B, \mathcal{O}_B(Np))) \\ s_t(B) &\mapsto s_t^*\mathcal{D}. \end{aligned}$$

S -integral points are sections s_t mapping to elements in $\Gamma(B, \mathcal{O}_B(\mathcal{D}))$ vanishing at p to maximal order N . Thus we are interested in elements of $G^{-1}[\Gamma(B, \mathcal{O}_B)]$, where $\Gamma(B, \mathcal{O}_B) \subset \Gamma(B, \mathcal{O}_B(Np))$ corresponds to the constant functions, i.e., the image of the map on global sections induced by the inclusion of sheaves

$$\mathcal{O}_B \hookrightarrow \mathcal{O}_B(Np).$$

The indeterminacy of G is resolved by blowing up the subscheme (5.1). The stalk of its normal bundle at $s(B)$ is canonically isomorphic to $\Gamma(\mathcal{O}_{\mathcal{X}}(\mathcal{D})|s(B))$. In particular, the proper transform of $G^{-1}[\Gamma(B, \mathcal{O}_B)]$ meets the exceptional fiber over $s(B)$ at the point

$$[\Gamma(B, \mathcal{O}_B)] \in \mathbb{P}(\Gamma(\mathcal{O}_{\mathcal{X}}(\mathcal{D})|s(B))) \simeq \mathbb{P}(\Gamma(B, \mathcal{O}_B(Np))).$$

Thus $s(B)$ deforms to $s_t(B) \in G^{-1}[\Gamma(B, \mathcal{O}_B)]$, corresponding to an S -integral point.

The sections thus produced are Zariski dense in \mathcal{X} . Indeed, our construction produces sections passing through the generic point of \mathcal{D} that deform out of \mathcal{D} to the generic point of \mathcal{X} .

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